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Research Article

Low Energy Nuclear fusion with Two Photon Emission

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Abstract

We consider a recently proposed mechanism for low energy nuclear reactions (LENRs). In this mechanism the process is initiated by an external photon whose contribution is treated perturbatively. In the present paper we consider a related process that is induced by photon emission rather than absorption. We consider, as an example, fusion of hydrogen and deuterium to form He(3). This process, at leading order, proceeds by emission of a photon. In the present case we consider two photon emission and hence we need to go to the second order in perturbation theory. The emission of a high energy photon from the initial D-H system leads to the formation of an intermediate state which is a superposition of all energy eigenstates of the unperturbed Hamiltonian. We compute the cross section for such a reaction and show that, for suitable conditions in condensed matter medium, the reaction rate can be sufficiently large to be observable.
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1. Introduction

In a recent paper [1] (Paper I hereafter) the authors proposed a new mechanism which may be applicable to nuclear reactions at very low energies, on the order of 1 eV or lower. In paper I it was argued that although the standard two-body nuclear fusion reaction has a very small cross section at low energies, a related process which is induced by a photon present in the initial state may have a much larger cross section. Although the idea is applicable for a wide range of reactions, in [1] the authors considered a specific reaction

$$H\,(1) + D\,(2) + \gamma(\omega_i) \rightarrow He\,(3) + \gamma(\omega_f) \tag{1}$$

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Here a H(1) nucleus undergoes fusion with D(2) nucleus in order to form He(3). The process is assisted by the presence of a photon in the initial state. It was argued that the energy of this photon need not be very large and in [1] the authors took it to be in the infrared regime. Due to the presence of this photon the leading order contribution to this process is obtained at the second order in perturbation theory. This is in contrast to the standard fusion reaction [2]–[6] for these nuclei,

$$H\left(1\right)+D\left(2\right)\rightarrow He\left(3\right)+\gamma\left(\omega_{f}\right) \tag{2}$$

which gets dominant contribution at first order in perturbation theory. In [1] it was argued that the standard process dominates at energies on the order of 1 keV or higher. However, as the initial state energy becomes very small E$<<$ 1keV the cross section for Eq. (2) falls very sharply. In contrast the reaction shown in Eq. (1) may have a relatively mild dependence on energy and hence its cross section at low energies may exceed that of reaction in Eq. (2) by many orders of magnitude. The enhancement was found to be sufficiently large that it makes the phenomenon of low energy nuclear reactions observable in the laboratory as claimed by experimentalists over many years [7]–[27]. The photon induced process goes through by a non-resonant mechanism at second order in the perturbation theory. If the initial state photon has sufficiently high energy, on the order of 1 mega-electron volts, then it can be absorbed by an initial state nucleus and induce a transition to a high energy state which may facilitate nuclear fusion. However, in [1] the authors were interested in a low energy photon such that a transition to a high energy state is not possible. As the initial two particle state absorbs a photon it goes into a state which is not an eigenstate of the unperturbed Hamiltonian. We can write it as a linear superposition of all these eigenstates. This sum includes eigenstates of very high energy. For the fusion process we need to include contributions from all such eigenstates. Due to the high energy involved, the barrier penetration factor is relatively mild in this case. Hence this has the potential to lead to a considerable enhancement of cross section of these processes. At low energies the cross section is found to be many orders of magnitude higher in comparison to the leading order process, Eq. (2) [1]. In the current paper we critically examine this process and address several issues which were not properly resolved in [1]. Ideas somewhat similar to this have been proposed earlier [28]–[31]. However, the photon induced process was not discussed in these papers. Some other theoretical ideas which attempt to solve this problem are described in [32]–[37].

In the present paper we also consider the related process of emission of two photons. As we shall show this process is simpler and dominates over the one considered in [1] for most situations encountered in the laboratory. This is because it does not require an incident photon flux. In [1] the authors had assumed that the initial state is a free particle state with kinetic energy of 10 eV when the two nuclei are very far apart. In the present paper we shall also consider a bound state which is taken to be the ground state of the hydrogen-deuterium system. The formalism is general and can be applied to either case. This particular state is being considered for a sample calculation. As we shall see the process works only inside condensed matter medium. In our paper, however, we shall first ignore the medium effect. This will help us understand why it fails and what features are required in the medium to make it work.

2. Photon induced nuclear reactions

In this section we review the phenomenon of low energy nuclear reactions induced by a real photon [1]. We shall also generalize the formalism to the case of arbitrary atomic numbers for the nuclei participating in these reactions. Furthermore, we set up the calculation so that it can be performed directly by starting from a potential model. In [1] such details were not presented. As explained in [1] the process can also be induced by other particles such as an electron or a proton which will involve exchange of a virtual photon with the initial state nuclei. Here we consider only the case of a free photon in the initial state. Let X_1 and X_2 be two initial state nuclei with atomic numbers Z_1 and Z_2 respectively. The corresponding mass numbers are A_1 and A_2. The final state nucleus is denoted by X_3 and has

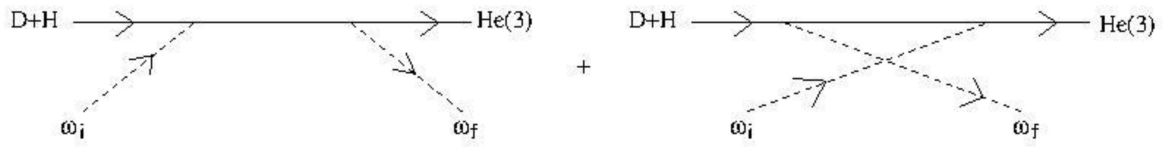

Figure 1. The fusion process initiated by absorption of an incident photon.

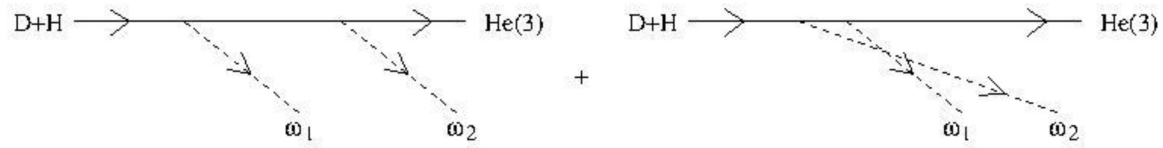

Figure 2. The fusion process initiated by emission of a photon from the initial state.

atomic and mass number Z and A respectively. We can express this process as (see Fig. 1)

$$X_1 + X_2 + \gamma(\omega_i) \rightarrow X_3 + \gamma(\omega_f) \tag{3}$$

Here, as in [1], we have assumed the presence of a photon in the final state. Other final states can also be considered but we will postpone this to future research.

In the present paper we shall also consider a related process which involves emission of two photons, i.e.

$$X_1 + X_2 \rightarrow X_3 + +\gamma\left(\omega_1\right) + \gamma(\omega_2) \tag{4}$$

This process is illustrated in Fig. 2. Both processes proceed through the formation of an intermediate state, however, the second process does not require an incident flux of photons. In the rest of this section we shall focus on the process given in Eq. (3). The other process will be considered in the next subsection.

Let *H* denote the Hamiltonian of the system. It is given by

$$H = H_0 + H_I \tag{5}$$

where H_0 denotes the unperturbed Hamiltonian and H_I is a time dependent perturbation. Let K_1 and K_2 be the kinetic energies of the two nuclei and *V*(r) be the effective potential. Here we will work in the centre of mass coordinate system and are interested only in the relative motion. The relative coordinate between the two nuclei is given by $\vec{r} = \vec{r}_2 - \vec{r}_1$ where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the two particles. The unperturbed Hamiltonian is given by

$$H_0 = K_1 + K_2 + V(r) \tag{6}$$

The effective potential includes the Coulomb repulsion between nuclei as well as the nuclear attraction. The screening effects [38]–[40] due to electrons are included by introducing a screening length in the Coulomb potential which should be sufficiently reliable for our purpose. Hence, we may express it as

$$V(r) = \frac{Z_1 Z_2 e^2}{r}\, e^{-r^2/r_0^2} + V_n + V_m \tag{7}$$

where V_n is the nuclear potential and V_m the molecular potential. The nuclear potential, for light nuclei, may be taken as

$$V_n = -V_0 e^{-r^2/R^2} \tag{8}$$

where V_0 is on the order of 50 MeV and R on the order of 2 fm. The precise values depend on nucleus and may be adjusted to fit data. In Eq. (7) we have modelled electron screening effects by introducing a screening length r_0 which may be taken to be approximately equal to the Bohr radius. Note that we have chosen a different form of the screening potential in comparison to what is usually chosen in the literature [38]. This is because we also have the Morse potential, described below, and we need the combination of the two to give the right molecular state. We shall adjust the parameters of the Morse potential slightly to achieve this. Note that these details have negligible effect on our results, especially since here we are interested only in demonstrating that low energy fusion is possible and not in precise numerical values. Hence, we are only interested in order of magnitude estimates for which our potential is sufficiently reliable.

The potential V_m for molecular bound state may be taken to be the Morse potential,

$$V_m = D\,\left(e^{-2a(r-r_m)} - 2e^{-a(r-r_m)}\right) \tag{9}$$

This potential is being used to obtain the molecular bound state. We have verified that it has negligible effect on the rest of the calculations, such as the computation of the intermediate states (discussed below). Hence for those calculations it will not be included. As we shall also argue the medium effects play an important role in the computation of intermediate states and hence the effective potential for these states is much more complex in comparison to the Morse potential.

The interaction Hamiltonian [41], [42] is given by

$$H_I\left(t\right) = -\frac{Z_1 e}{m_1 c}\,\vec{A}\left(\vec{r}_1, t\right)\cdot\vec{p}_1 + \frac{Z_2 e}{m_2 c}\,\vec{A}\left(\vec{r}_2, t\right)\cdot\vec{p}_2 + \frac{e\hbar g_p}{2m_p c}\,\vec{\sigma}\cdot\vec{B} + \ \ldots \tag{10}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of particle 1 and 2 respectively and the corresponding momenta $\vec{p}_1 = -\vec{p}_2 = -\vec{p}$. Here particles 1 and 2 refer to H and D respectively. The third term on the right-hand side corresponds to magnetic dipole moment of proton and g_p is the corresponding g factor. There is an additional contribution in Eq. (10) due to the deuteron magnetic moment, that we have not included in this expression. We will drop it here since its contribution is expected to be small. The electromagnetic field may be expressed as [42],

$$\vec{A}\left(\vec{r}, t\right) = \frac{1}{\sqrt{V}}\sum\nolimits_{\vec{k},\beta} c\,\sqrt{\frac{\hbar}{2\omega}}\left[a_{\vec{k},\beta}\,\vec{\varepsilon}_\beta e^{i\vec{k}\cdot\vec{r}-i\omega t} + a^\dagger_{\vec{k},\beta}\,\vec{\varepsilon}_\beta e^{-i\vec{k}\cdot\vec{r}+i\omega t}\right] \tag{11}$$

Here $a_{\vec{k},\beta}$ and $a^\dagger_{\vec{k},\beta}$ are the annihilation and creation operators, $\omega = |\vec{k}|$/c, c is the speed of light and $\vec{\varepsilon}_\beta$ the photon polarization vector.

The leading order contribution to the process given in Eq. (3) is obtained at second order in the time dependent perturbation theory. The transition amplitude at this order can be expressed as [41], [42],

$$\left\langle f \mid T\left(t, t_0\right) \mid i\right\rangle = \left(-\frac{i}{\hbar}\right)^2 \sum\nolimits_n \int_{t_0}^{t} dt' e^{i(E_f - E_n)t'/\hbar}\left\langle f \mid H_I\left(t'\right) \mid n\right\rangle \int_{t_0}^{t'} dt'' e^{i(E_n - E_i)t''/\hbar}\left\langle n \mid H_I\left(t''\right) \mid i\right\rangle \tag{12}$$

Here for brevity, we have not explicitly shown the photons present in the initial and final states. We point out that the final state nucleus is He(3) and has spin 1/2. We take the initial state to be also j = 1/2. The intermediate state $|n\rangle$ can have j = 1/2, 3/2 [1]. This is because it has orbital angular momentum l = 1. We next evaluate the integral over t′. The amplitude gets two contributions [42] which are shown in Fig. 1.

We next compute the amplitude corresponding to the first diagram shown in Fig. 1. We shall work in the dipole approximation and set $e^{-i\vec{k}\cdot\vec{r}_1} \sim e^{-i\vec{k}\cdot\vec{r}_2} \sim 1$. The t′′ integral gives [41], [42]

$$\int_{t_0}^{t'} dt'' e^{i(E_n-E_i)t''/\hbar} \langle n \mid H_I(t'') \mid i\rangle \;=\; -\frac{i\hbar e\xi}{\mu}\sqrt{\frac{\hbar}{2\omega_i V}}\,\langle n \mid \vec{\varepsilon}_\beta\cdot\vec{p} \mid i\rangle\,\frac{e^{i(E_n-E_i-\hbar\omega_i)t'/\hbar} - e^{i(E_n-E_i-\hbar\omega_i)t_0/\hbar}}{E_n-E_i-\hbar\omega_i} \tag{13}$$

where $\mu = \mathrm{m_1\ m_2/(m_1 + m_2)}$ is the reduced mass of the two-particle system and ξ is given by,

$$\xi = \frac{Z_2 m_1 - Z_1 m_2}{m_2 + m_1} \tag{14}$$

In our case $\mathrm{Z_1 = Z_2 = 1}$, $\mathrm{m_2}$ and $\mathrm{m_1}$ are the masses of hydrogen and deuterium respectively and hence $\xi = 1/3$. As explained in [1], the second term in Eq. (13), which depends on $\mathrm{t_0} \to \propto$, can be neglected [41, 42]. Here we focus on the first term.

We next evaluate the matrix element $\langle f|H_I(t')|n\rangle$ and substitute it along with the integral over t′′ in Eq. (12). We then obtain the following results for the transition matrix element [1]:

$$\langle f \mid T\,(t,t_0) \mid i\rangle = \mathrm{i}\frac{e^2\xi^2}{2\mu^2 V}\sqrt{\frac{1}{\omega_i\omega_f}}\int_{t_0}^{t} dt' e^{i(E_f-E_i-\hbar\omega_i+\hbar\omega_f)t'/\hbar}\sum\nolimits_n\left[\frac{\langle f \mid \vec{\varepsilon}_{\beta'}\cdot\vec{p} \mid n\rangle\,\langle n \mid \vec{\varepsilon}_\beta\cdot\vec{p} \mid i\rangle}{E_n-E_i-\hbar\omega_i}\right] \tag{15}$$

We point out that the second amplitude in which a photon is first created and later annihilated would involve the energy denominator $(E_n - E_i + \hbar\omega_f)$ instead of $(E_n - E_i - \hbar\omega_i)$. To proceed further we replace [1] the operator $\vec{p}$ in terms of the commutator of $\mathrm{H_0}$ and $\vec{r}$ using

$$\langle n \mid \vec{\varepsilon}_\beta\cdot\vec{p} \mid i\rangle = \frac{i\mu}{\hbar}\,(E_f - E_i)\,\langle n \mid \vec{\varepsilon}_\beta\cdot\vec{r} \mid i\rangle \tag{16}$$

We obtain,

$$\langle f \mid T\,(t,t_0) \mid i\rangle = -\frac{i\,e^2\xi^2}{2V\,\hbar^2}\sqrt{\frac{1}{\omega_i\omega_f}}\int_{t_0}^{t} dt' e^{i(E_f-E_i-\hbar\omega_i+\hbar\omega_f)t'/\hbar}\mathcal{M} \tag{17}$$

where

$$\mathcal{M} = \sum\nolimits_n\left[\frac{\langle f \mid \vec{\varepsilon}_{\beta'}\cdot\vec{r} \mid n\rangle\,\langle n \mid \vec{\varepsilon}_\beta\cdot\vec{r} \mid i\rangle}{E_n-E_i-\hbar\omega_i} + \frac{\langle f \mid \vec{\varepsilon}_\beta\cdot\vec{r} \mid n\rangle\,\langle n \mid \vec{\varepsilon}_{\beta'}\cdot\vec{r} \mid i\rangle}{E_n-E_i+\hbar\omega_f}\right](E_f-E_n)\,(E_n-E_i) \tag{18}$$

Here we have also added the second contribution in which the time order of the photons is reversed, as mentioned earlier.

Using this we obtain the transition rate,

$$\frac{d\tilde{P}}{dt} = \frac{1}{\Delta T}\int dE_{\gamma f}\rho_f |\langle f \mid T\,(t,t_0) \mid i\rangle|^2 \tag{19}$$

where $E_{\gamma f} = \hbar\omega_f$ is the final state photon energy and ρ_f is the number density of photon states at energy $E_{\gamma f}$, given by [42],

$$\rho_\gamma = \frac{V\omega_f^2}{(2\pi)^3}\frac{d\Omega}{\hbar c^3} \tag{20}$$

The time integral is proportional to $\Delta T\ \delta(E_f - E_i - \hbar\omega_i + \hbar\omega_f)$ where $\mathrm{\Delta T = t - t_0}$ is the total time. This transition rate corresponds to incident photon flux density of c/V. Hence, we should divide by this flux density and multiply by

the experimental flux density per unit frequency interval F_γ and integrate over frequency ω_I [42]. We should also sum over the final photon polarizations and average over the initial photon polarizations. This leads to [1]

$$\frac{dP}{dt} = \frac{1}{\Delta T}\int d\omega_i F \frac{V}{c}\int dE'_\gamma\, \rho_\gamma\, |\langle f \mid T\,(t,t_0) \mid i\rangle|^2$$
$$= \frac{\alpha^2\xi^4}{\hbar^2 c^2}\int d\Omega d\omega_i F \frac{\omega_f}{\omega_i}\frac{1}{2}\sum_{\beta\beta'}|\mathcal{M}|^2 \tag{21}$$

We can use this directly to compute the transition rate for the case of an initial bound state. If the initial state is a free particle state, we can use the procedure described in [1] to obtain the cross section. This essentially summarizes the basic theoretical structure of [1].

We next point out several problems with the calculation presented in [1]. The calculation involves the following key concepts:

1. The nuclear matrix element, $\langle f|\vec{\varepsilon}_\beta\cdot\vec{r}|n\rangle$. This can be calculated reliably once the nuclear potential is specified. The potential is of course uncertain, but simple models exist which are expected to be reliable up to small corrections.
2. The matrix element $\langle n|\vec{\varepsilon}_{\beta'}\cdot\vec{r}|i\rangle$ between the initial state $|i\rangle$ and the intermediate state $|n\rangle$. We are interested in an intermediate state of relatively high energy on the order of a few kilo-electron volts while the initial state has very low energy on the order of 1 eV. Hence this matrix element is expected to be very small and needs careful computation. In [1] the authors assumed free space wave function for the initial state and simply imposed a sharp cutoff on the distance scale. A more careful calculation with a bound state reveals that this matrix element is, in general, much smaller than that obtained in [1].
3. The sum over intermediate state energies in M in Eq. (18). This sum was performed in [1] by going to the continuum limit. The resulting integral, which has an upper limit of infinity, involves an overlap of a very rapidly varying factor arising from $\langle n|\vec{\varepsilon}_{\beta'}\cdot\vec{r}|i\rangle$ and a slowly varying factor due to the nuclear matrix element. In [1] this was computed using a finite upper limit on energy. However, a more detailed calculation involving a much higher upper limit reveals that this integral was also overestimated in [1].

Besides the problems mentioned above, an important point missed in [1] is that the second amplitude in Eq. (18) cannot be computed reliably in the dipole approximation. This is because the matrix element $\langle n|\vec{\varepsilon}_{\beta'}\cdot\vec{r}|i\rangle$ involves emission of a very high energy photon. For the present case, the energy is on the order of 5 MeV. Hence its wavenumber $|\vec{k}|$ is very large compared to the inverse of the atomic length scale and the exponential factor in the photon field, $e^{i\vec{k}.\vec{r}}$ cannot be set to unity. This factor turns out to be a very rapidly varying function of $\vec{r}$and offers the possibility of a much better overlap between states $|i\rangle$ and $|n\rangle$ in comparison to what is obtained by setting the exponential factor to unity. This in fact provides an elegant solution to the problem mentioned in [2] above. This also suggests that we might just consider the process in which two photons are emitted, one of which has a relatively high energy and a photon in the initial state is not required. We next turn to the two-photon emission process.

2.1. Transition rate for emission of two photons

In this section we determine the transition rate for the process given in Eq. (4). Although the calculation can be applied to many processes, here we shall focus on the case in which X_1 is a hydrogen, X_2 a deuterium and X_3 is He(3). We take the initial and final states to have orbital angular momentum, $l = 0$. The intermediate state has $l = 1$. Here we shall consider contribution from the magnetic moment interaction for the matrix element $\langle n, k_1|H_I|i\rangle$. Other terms

give similar contributions. We obtain,

$$\langle n, k_1 \mid H_I(t'') \mid i\rangle = \frac{ie\hbar g_p k_1}{2m_p\sqrt{V}}\sqrt{\frac{\hbar}{2\omega_1}}\left\langle n \left| \vec{\sigma}.\left(\hat{k}_1 \times \vec{\varepsilon}_\beta\right) e^{-i\vec{\mathrm{k}}_1.\vec{\mathrm{r}}_1 + i\omega_1 t''} \right| i\right\rangle \tag{22}$$

where $\mathrm{k}_1 = \omega_1/\mathrm{c}$ is the wave number of the photon created in this process. Here we have used

$$H_I(t) = \frac{e\hbar g_p}{2m_p c}\,\vec{\sigma}\cdot\vec{B} + \cdots$$

since the remaining terms in $\mathrm{H_I}$ are sub-dominant. We also have,

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{i}{\sqrt{V}}\sum\nolimits_{\vec{k},\beta} c\sqrt{\frac{\hbar}{2\omega}}\left[a_{\vec{k},\beta}\,\vec{k}\times\vec{\varepsilon}_\beta e^{i\vec{k}\cdot\vec{r}-i\omega t} - a^{\dagger}_{\vec{k},\beta}\vec{k}\times\ \vec{\varepsilon}_\beta e^{-i\vec{k}\cdot\vec{r}+i\omega t}\right]$$

Only the second term proportional to the creation operator contributes in Eq. (22). We replace the coordinate $\vec{\mathrm{r}}_1$ in terms of the relative coordinate $\vec{r}$ in Eq. (22) using,

$$\vec{\mathrm{r}}_1 = -\vec{r}\frac{m_2}{M} + \vec{\mathrm{R}}_{cm} \tag{23}$$

where $\mathrm{M} = \mathrm{m}_1 + \mathrm{m}_2$ and $\vec{\mathrm{R}}_{cm}$ is the center of mass coordinate. The center of mass coordinate is not relevant for this, so we focus on the relative coordinate. The initial and final states have $\mathrm{s} = 1/2, \mathrm{l} = 0$ and $\mathrm{j} = 1/2$. We consider the initial state with $\langle j_z\rangle = -\hbar/2$. The final state would have $\mathrm{s} = 1/2, \mathrm{l} = 0, \mathrm{j} = 1/2$ and $\langle j_z\rangle = \hbar/2$. The initial state corresponding to $\langle j_z\rangle = \hbar/2$ will make a transition to a final state $\langle j_z\rangle = -\hbar/2$ and will lead to the same reaction rate. The total rate is obtained by averaging over the two.

The intermediate state has orbital angular momentum $\mathrm{l} = 1$ and hence can have $\mathrm{j} = 1/2$ or $3/2$. The states that can contribute are,

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2};1,1\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle \tag{24}$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2};1,1\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2};1,0\right\rangle \tag{25}$$

Here on the left-hand side we have used the notation $|j, m_j\rangle$ where $\langle j_z\rangle = \hbar m_j$ and on the right-hand side the states are given in notation $|s, m_s; l, m_l\rangle$. We need to add the amplitudes corresponding to both states. We point out that the unperturbed energies of these two states are the same since both correspond to $\mathrm{l} = 1$. Let us consider a coordinate system in which the wave vector $\vec{\mathrm{k}}_1$ points along the z direction. The corresponding photon polarization vectors can be chosen to point along the x and y axis i.e., $\vec{\varepsilon}_1 = \hat{x}$ and $\vec{\varepsilon}_2 = \hat{y}$. We replace the Pauli matrices in terms of the spin matrices i.e., $\vec{s} = \vec{\sigma}\hbar/2$. Hence $\vec{s}\cdot(\hat{k}_1\times\vec{\varepsilon}_1) = s_y$. We can now replace this in terms of spin up and spin down operators, $\mathrm{s}_\pm$. For our case only the spin up operator contributes.

Using this we can compute the amplitude in Eq. (22). Let us consider the contribution from the intermediate state $|\frac{3}{2},\frac{1}{2}\rangle$ and polarization vector $\vec{\varepsilon}_1$. Only the second state, $|\frac{1}{2},\frac{1}{2};1,0\rangle$, in Eq. (24) contributes. Since this state has $\mathrm{l_z} = 0$ the angular part of the wave function is proportional to the spherical harmonic, ${\mathrm{Y}_1}^{0*}$. We obtain,

$$\langle n, k_1 \mid H_I(t'') \mid i\rangle = \frac{eg_p k_1}{2m_p\sqrt{V}}\sqrt{\frac{\hbar}{2\omega_1}}\sqrt{\frac{2}{3}}\int d^3r R_n^* {Y_1}^{0*} e^{-i\vec{\mathrm{k}}_1.\vec{\mathrm{r}}_1 + i\omega_1 t''} R_i Y_0^0 \left\langle\frac{1}{2},\frac{1}{2}\right| s_+ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \tag{26}$$

Here $\mathrm{R_i}$ and $\mathrm{R_n}$ are the initial and intermediate state radial wave functions that depend only on r. Besides this term we also get a contribution from the $\mathrm{j} = 1/2$ intermediate state. We now insert the explicit form of the spherical

harmonics and perform the angular integrals. The exponent in Eq. (26) is equal to $(-ik_1 r m_2 cos\theta/M)$. We define,

$$\tilde{k}_1 = k_1 \frac{m_2}{M}$$

In our case, m_2 /M = 2/3. This leads to

$$\langle n, k_1 \mid H_I(t'') \mid i\rangle = \frac{ieg_p k_1 \hbar}{2m_p\sqrt{V}}\sqrt{\frac{3\hbar}{2\omega_1}}\sqrt{\frac{2}{3}}\int drr^2 R_n^* \frac{\cos\tilde{k}_1 r}{\tilde{k}_1 r} R_i e^{i\omega_1 t''} \tag{27}$$

The matrix element in Eq. (27) contains one of the important concepts of this paper. The energy of the emitted photon E_1 can be as high as 5.4 MeV. This leads to a relatively large wave number $k_1 = E_1/(\hbar c)$ for the photon. We are interested in intermediate states with energies on the order of 1 keV or higher. These wave functions also have a sinusoidal behaviour, i.e. $u_n \sim \cos(k_n r)$. The product of the two cosine functions leads to

$$cos\,(k_n r)\,cos\left(\widetilde{\mathrm{k}}_1 r\right) = \frac{1}{2}\left(cos\left[\left(k_n - \widetilde{\mathrm{k}}_1\right) r\right] + cos\left[\left(k_n + \widetilde{\mathrm{k}}_1\right) r\right]\right) \tag{28}$$

Due to its low energy, the initial state wavefunction has a relatively slow dependence on r. Hence it is clear that in the neighbourhood of $k_n \sim \widetilde{\mathrm{k}}_1$ the matrix element can get significant contributions from the first term on the right-hand side of the above equation. The second term, however, contributes negligibly due to very rapid oscillations and can be dropped

In order to compute the nuclear matrix element, $\langle f, k_1, k_2|H_I|n, k_1\rangle$, we keep only the first two terms on the right-hand side of Eq. (10). The remaining magnetic moment term is also expected to contribute. Its contribution is expected to be smaller but not negligible. For our order of magnitude estimate and for the purpose demonstrating that this mechanism is possible, we can ignore this contribution. This amplitude involves creation of an additional photon of wave vector $\vec{k}_2$. We take the direction of this vector to be,

$$\hat{k}_2 = \sin\theta_2\,(\cos\phi_2\hat{x} + \sin\phi_2\hat{y}) + \cos\theta_2\hat{z} \tag{29}$$

The corresponding polarization vectors can be expressed as

$$\vec{\varepsilon}_1' = \cos\theta_2\,(\cos\phi_2\hat{x} + \sin\phi_2\hat{y}) - \sin\theta_2\hat{z} \tag{30}$$

$$\vec{\varepsilon}_2' = -\sin\phi_2\hat{x} + \cos\phi_2\hat{y} \tag{31}$$

For this matrix element we can use the dipole approximation and set the factor $\exp(-i\vec{k}_2\cdot\vec{r})$ in the photon field approximately equal to 1. Furthermore, only the polarization vector $\vec{\varepsilon}_1'$ contributes. We obtain

$$\langle f, k_1, k_2 \mid H_I(t') \mid n, k_1\rangle = \frac{ie\xi}{\hbar\sqrt{V}}\sqrt{\frac{\hbar}{6\omega_2}}\sqrt{\frac{2}{3}}\,(E_f - E_n)\sin\theta_2\int dr' r'^2 R_f^* r' R_n e^{i\omega_1 t'} \tag{32}$$

where R_f is the final state radial wave function. We also need the contribution from j = 1/2 intermediate state. This gives the same results as in Eq. (27) and (31) with the factors $\sqrt{2/3}$ replaced by $\sqrt{1/3}$. Hence in the transition amplitude these two states, j = 3/2 and 1/2 lead to factors 2/3 and 1/3 respectively. Adding their contributions just leads to a factor of unity.

The transition matrix element, for the photon polarization vectors $\varepsilon_1, \vec{\varepsilon}_1'$ can be written as

$$\langle f, k_1, k_2 \mid T \mid i\rangle = -\frac{ie^2\xi g_p k_1}{4Vm_p\sqrt{\omega_1\omega_2}}\sin\theta_2\int_{t_0}^{t} dt' e^{i(E_f - E_i + E_1 + E_2)t'/\hbar}\sum_n \frac{(E_f - E_n)}{E_n - E_i + \hbar\omega_2} I_2 I_1 \tag{33}$$

Here $E_1 = \hbar\omega_1, E_2 = \hbar\omega_2$,

$$I_1 = \int_0^\infty drr^2 R_n^* \frac{\cos\tilde{k}_1 r}{\tilde{k}_1 r} R_i \tag{34}$$

and

$$I_2 = \int_0^{\propto} dr' r'^2 R_f^* r' R_n \tag{35}$$

Furthermore, we have performed an integral over t′′ and dropped the term dependent on t_0 [1]. A small contribution due to recoil of the nucleus is ignored in the energy denominator. The limits on the integral over t′ has to be taken to $\pm\propto$ at the end of the calculation. We also set the total time $t - t_0 = \Delta T$. The transition rate can now be computed inserting the number density of photon states,

$$\rho_\gamma = \frac{V\omega_f^2}{(2\pi)^3}\frac{d\Omega}{\hbar c^3}$$

and integrating over the photon energies. In the present case we have two photons in the final state with frequencies ω_1 and ω_2. Hence, we need to insert number densities for both of these. We obtain,

$$\frac{dP}{dt} = \frac{1}{\Delta T}\int dE_1 dE_2 \frac{V\omega_2^2}{(2\pi)^3}\frac{d\Omega_2}{\hbar c^3}\frac{V\omega_1^2}{(2\pi)^3}\frac{d\Omega_1}{\hbar c^3}|\langle f, k_1, k_2 \mid T(t,t_0) \mid i\rangle|^2 \tag{36}$$

Here the integral is over the energies E_1, E_2 and the solid angles Ω_1 and Ω_2. We obtain,

$$\frac{dP}{dt} = \frac{\alpha^2\xi^2 g_p^2}{32\pi^3 m_p^2\hbar^3 c^6}\int dE_1 dE_2 d\Omega_2 d\Omega_1 sin^2\theta_2 E_1^3 E_2 \left|\sum_n \frac{(E_f - E_n)}{E_n - E_i + \hbar\omega_2} I_2 I_1\right|^2 \tag{37}$$

Performing the angular integrals, we obtain,

$$\frac{dP}{dt} = \frac{\alpha^2\xi^2 g_p^2}{6\pi m_p^2\hbar^3 c^6}\int dE_1 dE_2\ E_1^3 E_2 \tag{38}$$

We express the radial wave functions as,

$$R_i = \frac{u_i(r)}{r} \tag{39}$$

$$R_n = \frac{u_n(r)}{r} \tag{40}$$

$$R_f = \frac{u_f(r)}{r} \tag{41}$$

The wave function u_i is normalized such that,

$$\int_0^{\propto} dr\ u_i^* u_i = 1,$$

along with a similar equation for u_f. The wave function u_i is shown in Fig. 3. The parameters chosen in obtaining this wave function are given in the next section. The intermediate state wave function $u_n(r)$ behaves as $cos(k_n r + \delta)$ at large r. Here δ is an energy and r dependent phase shift. It is a slowly varying function, and the dominant behaviour is sinusoidal, $u_n \sim \sin(k_n r)$ or $u_n \sim \cos(k_n r)$, i.e., we can express u_n in terms of these functions with slowly varying coefficients. It is normalized over a length scale L, which is taken to infinity at the end of the calculation. In this limit, the sum over n in Eq. (37) converts into an integral and the factors of L cancel out. We show numerical solution for $u_n(r)$ for $E_n = 1$ keV in Fig. 4. In this figure the normalization factor has been set to unity.

We also need to sum over the final state photon polarizations, as well as average over initial nuclear spin states. The two spin states give the same contribution so averaging just leads to unity. For the photon polarizations, both

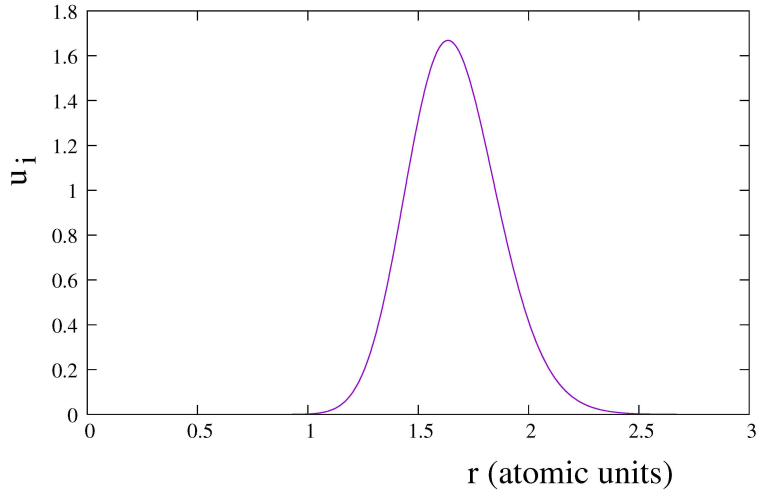

Figure 3. The initial state D-H ground state wave function.

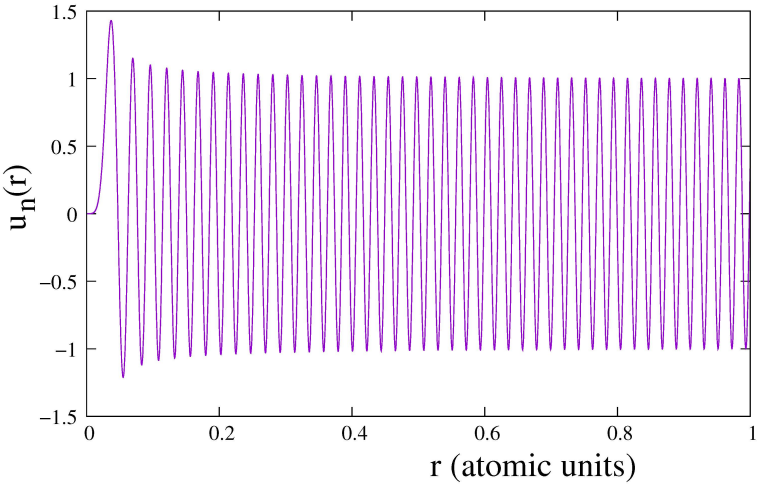

Figure 4. An intermediate state wave function u_n for $E_n = 1$ keV.

polarization vectors contribute equally for photon of energy E_1 while only one vector contributes for the second photon. Hence, we obtain an overall factor 2. We finally obtain,

$$\frac{dP}{dt} = \frac{\alpha^2 \xi^2 g_p^2}{3\pi\hbar^3 c^6 m_p^2} \int dE_1 E_1^3 E_2 |I|^2 \tag{42}$$

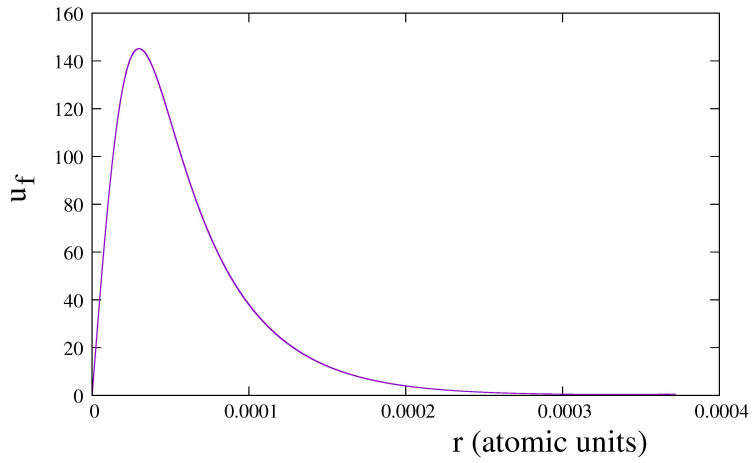

Figure 5. The final state He(3) nuclear ground state wave function.

where,

$$I = \sum_n I_1 I_2 \frac{E_f - E_n}{E_n - E_i + \hbar\omega_1} \tag{43}$$

Here the energies satisfy the relationship $E_i = E_f + E_1 + E_2$. We point out that $E_f = -5.47$ MeV. The sum in *I* may be evaluated by using the continuum limit and converting this to an integral.

3. Transition Rate for He(3) production

In this section we compute the transition rate for the process discussed in section 2.1. The calculation needs to be performed taking the medium effects into account. The medium leads to screening effects which are already incorporated in the potential given in Eq. (7). The medium also modifies the two-body wave functions. This modification of the intermediate state wave functions u_n is likely to play a very important role in our calculation, as we shall discuss. However before including medium effects it is useful to perform a free space calculation which provides useful insights. By free space we mean that the potential simply corresponds to effective potential of the H-D system, including screening effects.

The parameters of the potential are taken to be:

- Morse Potential: $D = 0.2$, $a = 1.01$ and $r_n = 1.4$ in atomic units.
- Nuclear Potential: $V_0 = 55$ MeV, $R = 1.81$ fm.
- Screened Coulomb potential: $Z_1 = Z_2 = 1$, $r_0 = 0.8$ in atomic units.

We point out that the molecular potential is the combination of the Morse potential and the screened Coulomb potential. A detailed modelling of this potential can be done for precise calculations, which are not relevant for our order of magnitude estimates. Small changes in this potential play a negligible role in our final estimates. These parameters produce a D-H molecular ground state (Fig. 3) at energy -4.68 eV and nuclear ground state (Fig. 5) with binding energy -5.49 MeV, in agreement with data. Furthermore, we verified that the cross section for the standard

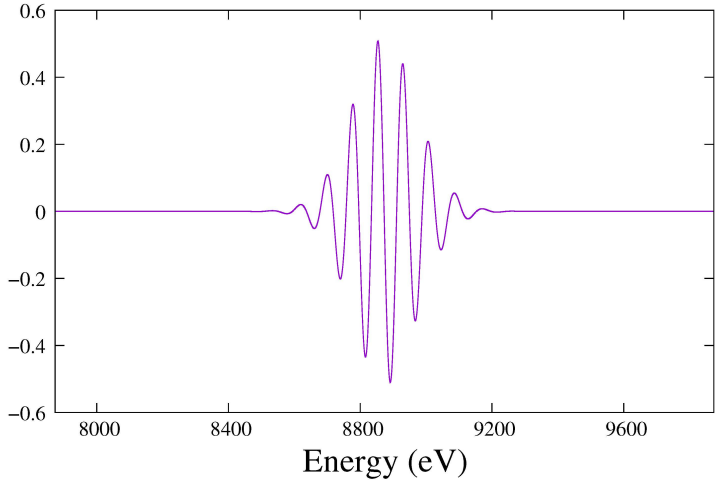

Figure 6. The integral $I_1 \times \tilde{k}_1$ as a function of the intermediate state energy $\mathrm{E_n}$.

process, given in Eq. (2), obtained using these parameters is in good agreement with data and the standard calculation at kilo-electron volt energies.

In Fig. 6 we show the integral $I_1 \times \tilde{\mathrm{k}}_1$ as a function of the energy $\mathrm{E_n}$. In this evaluation we have set the energy of photon $\mathrm{E}_1 = 5$ MeV. As expected, we find a large value in the neighbourhood of $k_n \sim \tilde{\mathrm{k}}_1$. The integral decays sharply to zero away from this interval. It also shows rapid oscillations. This implies that unless it is multiplied by another rapidly varying function the transition rate is likely to be relatively small. The integral I_2 indeed shows relatively slow variation over the narrow range in which I_1 takes significant values. Hence, we do expect a significant suppression of the transition rate. The quantity I which involves the sum over energies $\mathrm{E_n}$, computed in the continuum approximation, is shown in Fig. 7 as a function of the upper limit in energy. We find that the total amplitude becomes substantial for some values of the cut-off but eventually for larger values decays to very small values. It is interesting to determine exactly what the final value is, since even a contribution many orders of magnitude smaller than the peak value can lead to observable results. However, this requires us to perform a very detailed numerical calculation extending over an infinite range of energy eigenfunctions. Furthermore, experimental research suggests that the process requires special medium properties. Hence, we postpone a detailed energy integral in free space to future research. Here we suggest some alternatives which lead to a significant contribution directly from the region close to the peak in Fig. 7.

The result above suggests that using free space (including screening effects) wave functions $\mathrm{u_n}$ leads to a relatively small value of transition amplitude. This is perhaps in agreement in experimental results which indicate that the process happens only in a medium. Furthermore, it is known that special medium properties are required to boost the process to an observable rate. In the next section we examine this in detail.

4. Requirements for Increasing the Transition Rate

In the previous section we found that the reaction rate is very small if we use the 'free space' wave functions for the intermediate states. These wave functions are obtained by assuming spherical symmetry and display sinusoidal behavior at large distances. We found that these lead to a significantly large value of the molecular matrix element, essentially the integral I_1 in Eq. (37), in a narrow range of intermediate state energies $\mathrm{E_n}$. However, once we sum

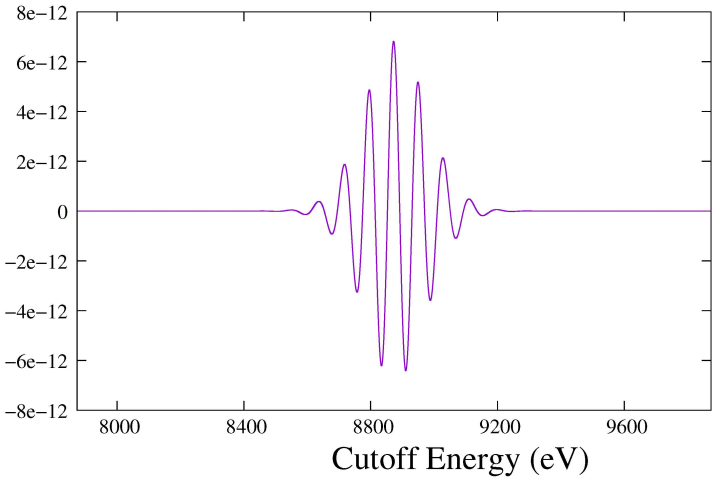

Figure 7. The sum over energies *I* evaluated in the continuum limit as a function of the upper limit on energy E_n.

over all energies, i.e., the sum *I* in Eq. (37), the total contribution adds up to relatively small values. This is shown in Fig. 7 as a function of the cutoff energy. We see that the integral does acquire significant values for some range of cutoff energies but eventually decays. The mathematical reason for this is easily understood. The sum (or integral) *I* involves a rapidly varying function of energy I_1 multiplied by a slowly varying nuclear matrix element I_2. The integral is essentially controlled by I_1 which leads to an oscillatory behavior with small amplitude at large energy cutoffs. In order to get a contribution near the peak in Fig. 6 we need an additional fast variation with energy, or we need a modification in the molecular matrix element. We next describe a few situations in which this can arise.

4.1. Resonant Tunnelling through Intermediate state

Let us assume that in the energy interval in which molecular matrix element I_1 takes significant values, there exists a state corresponding to resonant tunnelling with a relatively narrow width. The width may be on the order of 100 eV. In that case, as can be seen in Fig. 6 the cancellation will no longer arise and the integral I will take substantial values. For the present example of fusion of hydrogen with deuterium, this is likely to be the case as long as a resonance is present in the energy interval from 1 keV to about 10 keV. For lower energies the tunnelling barrier is too strong, and the upper limit is dictated by the maximum energy of the photon that can be emitted. Such a resonant state does not exist for the process under consideration. However, it is possible that it may exist in other cases. Furthermore, inside a medium the transition rate can be enhanced even if a narrow resonant state within this energy range is present in the lattice ion or an impurity ion in the medium. This is because, at that energy, the amplitude for this state will show a sharp change at the resonant energy.

4.2. Medium Properties

We next describe some situations in which the medium can play a crucial role in enhancing the reaction rate. We point out that this is indeed what is observed in experiments. The process does require special medium conditions in order to proceed at observable rates. We first note that the free space wave functions used in the previous section are simply invalid in medium in which the assumed spherical symmetry is not applicable. We point out that we are interested

in wave functions of relatively high energy on the order of 1 keV or higher. Nuclei of such high energy inside a medium are likely to significantly distort the medium itself and hence it may not be possible to reduce the problem to an effective background potential. Dealing with the full problem is clearly rather complicated, so we will not address this issue in this paper but will restrict ourselves to an effective potential. The additional effects can perhaps be treated perturbatively.

An important point is that in the medium the wave functions are likely to decay beyond a certain distance. In any direction the wave function would decay once it encounters an ion. In a periodic lattice, such wave functions might lead to a band structure. Here we are referring to band structure of light nuclei and not electrons. While it will be interesting to investigate this in detail, here we assume that the medium has enough impurities or is amorphous. Hence the band structure, even if it is present, would be significantly distorted. The main point that we wish to pursue here is the possible existence of localized states, analogous to Anderson localization [43, 44]. As long as such states exist with significant energy gaps, the cancellation seen in Fig. 7 would not happen, and we expect to get a significant contribution. Such states are likely to exist in the presence of disorder. If they exist, we can simply ignore all the other states and only include the contribution due to them. This requires careful investigation, and we hope to pursue this in a future publication. The possible existence of isolated states in a different framework has also been suggested in [31].

We also note that at some energies we expect to see threshold effects. This would arise if the energy $\mathrm{E_N}$ of the two-body system (D, H in the present case) is such that it can knockout an ion from the lattice or an inner electron from one of the atoms. At such thresholds we expect to see a sudden change in the nature of wave functions, which hence may lead to the sharp change as a function of energy required obtain a significant contribution.

An important property of the medium that we wish to exploit is the fact that it does not have the spherical symmetry of free space. Hence at distances much larger than the nuclear scale, the eigen-functions of the system are simply not the spherically symmetric states. These are controlled by the boundary conditions imposed by the medium. Given the right boundary conditions, we find that the reaction rate can be enhanced to very large values. We illustrate this explicitly in the next subsection by taking a simple example.

4.3. A Toy Model

In this section we present a toy model of medium wave function ψ_{n} which can lead to enhanced rates of the low energy nuclear reaction under consideration. Here we present only one example and do not attempt an exhaustive survey of medium properties that may be required. Our example makes use of the breakdown of spherical symmetry in the medium. Hence the wave function is not spherically symmetric at large distances, while at small nuclear scale distances this symmetry is maintained. The basic idea can be explained by referring to the molecular matrix element Eq. (24). If the two wave functions ψ_{i} and ψ_{n} behave essentially as plane waves pointing close to the direction of the emitted photon, then this matrix element will get a dominant contribution only from a very narrow range of frequencies. Hence in such a case we would not expect to see the cancellation that arose in the earlier case, and we may get larger contribution. We accomplish this by assuming cylindrical symmetry around the z direction. We just need a preferred direction; precise cylindrical symmetry is not really necessary but is convenient for calculations.

Consider a $\mathrm{D}-\mathrm{H}$ system inside a medium such that, at distances on the order of a few Bohr radii, the medium displays cylindrical symmetry (Fig. 8). Hence, we assume that the potential at relatively large distances has cylindrical symmetry, while at smaller distances it has the standard spherically symmetric behavior. The potential may be expressed as

$$V\left(z,\rho,\phi\right)=V(\rho,z) \tag{44}$$

at z and ρ large compared to nuclear scales. Along the z axis the potential is taken to be symmetric under $\mathrm{z}\rightarrow-\mathrm{z}$, for simplicity. We assume that the wave functions along z display free particle behavior up to a certain distance from

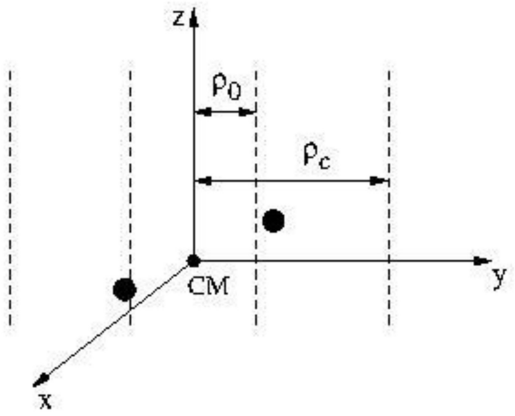

Figure 8. We assume that the D-H system is in a medium which displays cylindrical symmetry about the z-axis. Here the center of mass (CM) of the D-H system is taken to be at the origin of the coordinate system. The dominant contribution to the molecular matrix element comes from the region between $\rho_0<\rho<\rho_c$ where $\rho_0<a_0$ and $\rho_c \sim$ few a_0. In this gap region the potential is relatively weak and the wave functions display sinusoidal behaviour along the z direction.

the center of mass of the $D-H$ system and then decay due to Coulomb repulsion offered by the medium ions. The wave function would also decay as the D and H get close to one another. The energy of the initial state is taken to be on the order of 0.1 eV. Its wave function is assumed to extend from $-z_0$ to $+z_0$, where z_0 is taken to be much larger than a Bohr radius. The actual value does not matter as long as it is sufficiently large. The higher energy intermediate state wave functions would extend to larger distances, which we denote by L. The factor L would cancel out of our calculation.

We can now choose a suitable cylindrically symmetric potential and obtain the corresponding wave function. We postpone a detailed investigation with potential model to future research. Here we simply assume a reasonable form of the wave function. We assume that the intermediate wave function at large distances takes the form

$$\psi_n(z,\rho,\phi) = X_1(\rho)\, e^{ik_z z} + X_2(\rho)\, e^{-ik_z z} \tag{45}$$

where k_z is taken to be a large wave number of the order k_1, the wave number of the emitted photon. This form is applicable at distances that are large compared to nuclear distances. Hence this applies, as long as the distance between D and H as well as the distance between either D or H and a lattice ion is large compared to nuclear scales. As we get close to any ion the wave function would decay to zero.

Let the energy eigenvalue be E_n. Let the wave vector $k_n = \sqrt{2\mu E_n/\hbar^2}$ be such that $k_n \approx \tilde{k}_1$. We seek wave functions for which $k_z \approx \tilde{k}_1$. Physically this means that the wave function varies rapidly along the z-axis and slowly along the ρ-axis. Substituting either one of the two components in Eq. (45) in the Schrodinger equation, we obtain

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial X_i}{\partial\rho}\right) + \left(k_n^2 - k_z^2\right) X_i - \frac{2\mu}{\hbar^2}\, V(\rho)\, X_i = 0 \tag{46}$$

for $i = 1, 2$. Here we have assumed that the z dependence of the potential is negligible over the region of interest. For simplicity we also ignore the ρ dependence. Then the solution to this equation is the Bessel function $J_0(\overline{\rho})$ where $\overline{\rho} = \rho\sqrt{k_n^2 - k_z^2}$. Here we have assumed $k_n^2 \approx k_z^2$. For large ρ as we approach some lattice sites the wave function would decay and deviate from the pure Bessel behavior. Such details, as well as a suitable potential form over the entire region, can be easily incorporated in a detailed study but should not change our results qualitatively.

We need to now match this to the short distance solution which is dominated by the spherically symmetric Coulomb potential. The solution for the case of Coulomb potential is somewhat complicated and would require detailed analysis perhaps with the help of numerical integration. Here our main purpose is to illustrate that there is no problem in constructing such a solution. For this purpose, we shall use a square well potential. Hence at short distances, $r << a_0$, where a_0 is the Bohr radius, we assume the following form of the potential

$$\begin{aligned} V(r) &= -V_0 \quad r < r_0 \\ &= V_1 \quad r \geq r_0 < r_1 \\ &= 0 \quad r \geq r_1 \end{aligned} \tag{47}$$

in spherical polar coordinates. Here V_0 represents the attractive nuclear potential and V_1 is a model which replaces the repulsive Coulomb potential. Here we shall choose r_1 as the distance which corresponds to the turning point of the Coulomb potential for energy E_n. For $r < r_1$ we expand the wave function in spherical harmonics, that is,

$$\Psi\ (r, \theta, \varphi) = \ R_{n,l}\ (r)\ Y_l^m(\theta, \varphi) \tag{48}$$

We need to match this wave function to the large distance wave function at $r = r_1$. Note that since we have set the potential equal to zero for $r > r_1$, the wave function in cylindrical coordinates given earlier is applicable in this entire range. It is clear that there is no problem in matching the two wave functions at $r = r_1$. Hence, we expect that solutions of this kind exist.

Although the desired solution exists, we still need to show that the required $l = 1$ state gives substantial contribution at short distances, i.e., whether $l = 1$ gives significant contribution in Eq. (48). We can address this by considering the variation of the wave function ψ ($r > r_1$) in the vicinity of r_1. In this region $J_0(\rho) \approx 1$ and the solution is dominated by the z dependence. Let us determine the variation of the wave function at $r = r_1$ as we vary z from r_1 to 0. This would be equivalent to the change of polar angle from 0 to $\pi/2$. The wave function has the sinusoidal dependence given in Eq. (45). The wave number k_z corresponds to energy on the order of 1 keV to 10 keV. Let us consider the upper value. In atomic units this leads to a value $k_z \approx 900$. We next determine r_1 by taking this as the turning point of the Coulomb potential for energy 10 keV. We find this to be roughly 1/300 in atomic units. Hence as z changes from 0 to r_1, $k_z z$ changes from 0 to 3. Given the relatively slow change it is clear that the $l = 1$ component is expected to give a substantial contribution. Hence it is clear that the desired wave function exists.

In our calculation we shall replace the sum over intermediate energy eigenvalues E_n by an integral. This energy is approximately related to the wave number k_z in the z dependence of the wave function. Hence while computing the energy integral, we assume $E_n \sim \sqrt{2\mu E_n/\hbar^2}$. We can now replace the sum with an integral over k_z by setting

$$k_z\ L = 2n\pi \tag{49}$$

As already mentioned, L cancels out of our calculation.

For the initial state wave function, we assume a form similar to that in Eq. (45). In this case the wave number k_i would be relatively small. Hence, we take the form,

$$\Psi_i\ (z, \rho, \varphi) = AF(z)f(\rho) \tag{50}$$

where $f(\rho) \sim J_0(\widetilde{\rho})$, $\widetilde{\rho} = \tilde{k}_i \rho$ and $\tilde{k}_i$ is defined below. We will take the z dependence to be sinusoidal far from the ions. For ρ larger than about 0.5 Bohr radius, the Coulomb repulsion within the $D - H$ system would not be very effective, and we simply take the z dependence to be

$$F(z) = sin[k_{zi} z] \tag{51}$$

such that F(z) $\approx$ 0 for $z < -z_0$ and also for $z > z_0$. The wave function decays rapidly for $z < -z_0$ and $z > z_0$ due to the Coulomb repulsion from lattice ions. Here for simplicity, we simply set it to zero beyond this range. For smaller

ρ the z dependence is a little more complicated due to the strong Coulomb repulsion within the $D-H$ system. We will ignore this region in our integral. This is reasonable for our rough estimate. Let E_i be the energy eigenvalue for this wave function which is assumed to be on the order of 0.1 eV. Furthermore, we can obtain the value of $\tilde{k}_i$ in terms of E_i and k_{zi} from the Schrodinger equation. We emphasize that for ψ_i the sinusoidal z dependence is not necessary. It is being assumed only since it leads to convenient analytic computation. More general forms can be investigated in numerical analysis.

We next compute the transition rate with the wave function described above. Let us assume that the photon is emitted in the z direction. It need not be directly along the z-axis but can deviate a little from this direction, such that the transverse component of its wave vector is comparable to or smaller than $\tilde{k}_i$. This will lead to a suppression factor in the angular integration over the emitted photon directions of order $\left(\tilde{k}_i/\tilde{k}_1\right)^2$.

With these conditions we compute the molecular matrix given in Eq. (24). Both X_1 and X_2 are proportional to J_0. We may take the wave function of the form

$$\Psi_n = bg(\rho)\sin(k_n z + \Phi) \tag{52}$$

where $g(\rho) \sim J_0(\vec{\rho})$, Φ is a phase shift, which for a Coulomb potential depends on k and z. The molecular matrix element is given by

$$\left\langle n \left| e^{-i\vec{k}_1.\vec{r}m_2/M} \right| i \right\rangle = \int d^3r\, \Psi_n^* e^{-i\tilde{k}_1 z}\, \Psi_i \tag{53}$$

We now express the $\sin(k_n z+\Phi)$ as difference of two exponentials. The term involving $\exp(ik_n z + i\Phi)$ leads to a very rapidly oscillating function and hence gives negligible contribution. Keeping only the other term we obtain

$$\left\langle n \left| e^{-i\vec{k}_1\cdot\vec{r}m_2/M} \right| i \right\rangle = i\pi Ab \int d\rho\rho f(\rho)\, g(\rho) \int_{-z_0}^{z_0} dz \sin[k_{iz}z]\left[\cos(\Delta k_n z - \Phi) - i sin(\Delta k_n z - \Phi)\right] \tag{54}$$

where $\Delta k_n = k_n - \tilde{k}_1$. The integral over ρ extends from $\rho_0 \sim 0.5$ a_0, where a_0 is the Bohr radius, to $\rho_c \sim$ few a_0. This leads to a term of order unity. Here we ignore the contribution from ρ between 0 and ρ_0. In this region the wave function ψ_i gets significantly modified due to Coulomb repulsion within the $D-H$ system. This can easily be included by performing a more detailed numerical integral, but this approximation is acceptable for our order of magnitude estimate. The most interesting part of this is the integral over z. Both the real and imaginary part of this contributes independently and to demonstrate that this gives substantial contribution, we need to consider only one of these. Hence, we consider the integral

$$\begin{aligned} I_3 &= \int_{-z_0}^{z_0} dz \sin[k_{iz}z]\sin(\Delta k_n z - \Phi) \\ &= \int_{-z_0}^{z_0} dz(\cos[(\Delta k - k_{iz})z - \Phi] - \cos[(\Delta k_n + k_{iz})z - \Phi]) \end{aligned} \tag{55}$$

As expected, this integral takes significant values in the neighborhood of $|\Delta k_n| \sim k_i$. This works out to be,

$$I_3 = 2cos\Phi\left[\frac{\sin(\Delta k_n - k_{iz})z_0}{\Delta k_n - k_{iz}} - \frac{\sin(\Delta k_n + k_{iz})z_0}{\Delta k_n + k_{iz}}\right] \tag{56}$$

The other $\cos(\Delta k_n z - \Phi)$ term in Eq. (54) gives a similar result with an overall factor of $\sin\Phi$ instead of $\cos\Phi$. To get an order of magnitude estimate we drop the Φ dependence.

We can now determine the sum over all energies, i.e., the sum *I* given in Eq. (37), using the result for molecular overlap given in Eq. (55). Here we need to replace I_1 with I_3. We denote the nuclear matrix element I_2 by f(k). The

two terms in Eq. (55) can be approximated by delta functions in the limit of large z_0, which is applicable in the present case. Hence the integral over energies simply leads to the result

$$I \sim f\left(\tilde{k}_1 + k_{iz}\right) - f\left(\tilde{k}_1 - k_{iz}\right) \tag{57}$$

Taking all the factors into account, we may compute the transition rate for the process. We find this to be on the order of 10^{-16} per second for the process under consideration. The most interesting part of this result is that a large distance modification in the wave function produces a dramatically different result from that obtained in section 3. Here, by large distance, we mean distances on the order of the Bohr radius or larger. This clearly demonstrates the possibility that by suitable medium modifications it is possible to induce nuclear reactions, as often claimed by scientists working in low energy nuclear reactions.

There are several sources of uncertainty in this. The most important is our assumption of the medium wave function. In future, it will be useful to make a specific potential model in order to determine this. In particular, it is very important to study its matching with the short distance wave function where it is expected to display approximate spherical symmetry.

5. Experimental Test

Our prediction for the D-H fusion can be experimentally tested by observing the emitted photons. The Q value of this process is roughly 5.4 MeV. Hence, we predict that for each reaction two photons should be emitted in coincidence with their energy adding up to about 5 MeV, allowing for a small recoil energy of the nucleus. There exists some evidence that photons of relatively high energy may have been seen in low energy nuclear reactions [45]–[47]. However, a detailed test of our predictions is missing. Assuming 0.015% deuterium, applicable for naturally occurring hydrogen, we predict roughly 1000 events per second per cm^3. Here we have assumed a sample consisting of 10^{23} H atoms per cm^3. The observed γ ray energies would also need to be corrected for energy loss while propagating in medium.

6. Discussion and Conclusions

In this paper we have demonstrated that the mechanism for nuclear fusion introduced in [1] can be realized physically. In contrast to [1] we considered the process in which two photons are emitted rather than one being absorbed and another emitted. The latter process may dominate if the incident photon flux is very high. However, without creating special conditions in the laboratory, it is found that the process involving two photon emission dominates. The reaction proceeds by emitting a high energy photon and making a transition to an intermediate state which is not an eigenstate of the unperturbed Hamiltonian. The process requires either a resonant nuclear fusion transition at some high energy or a special wave function in the medium. Without this, the amplitude is found to be very small. We have given one example of the medium wave function that can lead to an observable reaction rate. Our result clearly shows that it is possible to tune medium properties in order to enhance the nuclear fusion rates by many orders of magnitude. Essentially, we have given a proof of principle that the phenomenon of low energy nuclear reactions can take place at observable rates.

The input energies required to initiate a nuclear reaction are on the order of 1 eV. Hence, we solve an important part of the LENR puzzle as listed, for example, in [48]. The reaction rate also depends on the value of the initial state energy and hence on the medium temperature, as observed in many LENR experiments. Furthermore, it is likely that there may exist different medium conditions that may provide a suitable environment for such reactions to occur. In our paper we have suggested a few possibilities. We have so far focussed only on the fusion in the H-D system due to its simplicity. In this case, fusion is initiated by electromagnetic perturbation. In order to get a complete picture, it is necessary to look at other reactions and also consider other possibilities, such as fusion induced by weak interactions.

The role of phonons may also be considered [36]. Although the energies involved with phonons are much smaller than the required nuclear energies, they could provide a contribution if the process proceeds through the mechanism proposed in our paper, i.e., emission of a photon with mega-electron volt energies.

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