J. Condensed Matter Nucl. Sci. 34 (2021) 102–109

Research Article

Nonrelativistic Reduction of the Phonon–nuclear Interaction for the One-pion Exchange Potential with Pseudovector Interaction

Peter L. Hagelstein*

Massachusetts Institute of Technology, Cambridge, MA, USA

Abstract

The lowest-order relativistic phonon-nuclear interaction derived from the many-particle Dirac model is expressed in terms of the Dirac $\boldsymbol{\alpha}$ and β matrices, which requires a relativistic formalism to evaluate. To obtain estimates for the strength of the phonon-nuclear matrix elements for applications it would be useful to have a nonrelativistic version of the interaction. For the special case of the one-pion exchange potential based on the spatial part of the pseudovector pion-nucleon interaction an appropriate nonrelativistic reduction is derived.

Keywords: Nonrelativistic reduction, One-pion exchange potential, Phonon–nuclear interaction, Pseudovector coupling

1. Introduction

At this point we have a model for the relativistic phonon–nuclear interaction [1] which in our view provides a foundation for many of the anomalies seen in experiments in our field [2,3]. To make quantitative predictions from the models we have been working on, we are going to need to evaluate phonon–nuclear matrix elements for a variety of different physical systems. As with nearly all realistic nuclear calculations, this quickly becomes very complicated – in part because modern nuclear potential models are complicated, and in part because the wave functions are also very complicated. In the future it seems likely that systematic calculations will be possible when the phonon–nuclear interaction is implemented in very powerful codes for nuclear structure and matrix elements. For now, the near-term goal is to develop an approximate interaction that captures long-range contributions, which we can use for simple estimates.

The long range interaction is dominated by the one-pion exchange contribution, which makes it a focus of our study in this work. In the early days of nuclear physics, this interaction was often modeled based on an underlying pseudoscalar nucleon-pion interaction [4,5], which produces the well known nonrelativistic one-pion exchange central and tensor potentials [6]. Were we to work with the relativistic version of the pseudoscalar interaction [7], we would conclude that no associated phonon–nuclear interaction arises, since the pseudoscalar interaction commutes with the partial Foldy-Wouthuysen transformation $\hat{S}$ operator.

*E-mail: plh@mit.edu.

Nearly all modern nuclear calculations are carried out making use of the chiral effective field theory potentials [8–10], for which the long range one-pion exchange potential is modeled making use of the spatial part of the pion-nucleon pseudovector interaction. For this interaction there occurs a non-zero contribution to the phonon–nuclear interaction. Of interest in this work is the reduction of the relativistic one-pion exchange potential to the phonon–nuclear interaction.

2. Model

We are interested in developing a nonrelativistic reduction of the lowest-order relativistic phonon–nuclear interaction making use of the relativistic one-pion exchange potential based on pseudovector coupling.

2.1. Phonon–nuclear interaction

From previous work we write for the two-body potential contribution to the lowest-order phonon–nuclear interaction [1]

$$\hat{H}_{\text{int}} = \frac{1}{2Mc}\sum_{j<k}\left[\beta_j\boldsymbol{\alpha}_j + \beta_k\boldsymbol{\alpha}_k, \hat{V}_{jk}\right]\cdot\hat{\mathbf{P}} \tag{1}$$

where M is the total mass of the nucleus, where $\hat{\mathbf{P}}$ is the center of mass momentum which in a molecule or solid is a phonon operator, and where the β and $\boldsymbol{\alpha}$ operators are single-nucleon Dirac operators

$$\beta = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}\end{bmatrix}, \qquad \boldsymbol{\alpha} = \begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & \mathbf{0}\end{bmatrix}, \tag{2}$$

where the protons and neutrons are modeled using a Dirac formalism. The commutator that appears in this interaction is defined for two general operators $\hat{A}$ and $\hat{B}$ according to

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{3}$$

2.2. One-pion exchange potential

For the one-pion exchange potential based on the spatial part of the pseudovector interaction we can write

$$\hat{V}_{jk} = -\left(\frac{f_{\pi NN}}{m_\pi}\right)^2(\boldsymbol{\tau}_k\cdot\boldsymbol{\tau}_j)(\gamma_5\boldsymbol{\alpha})_k\cdot\int\frac{\mathbf{k}\mathbf{k}}{(m_\pi c^2)^2+\hbar^2c^2|\mathbf{k}|^2}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_k-\mathbf{r}_j)}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\cdot(\gamma_5\boldsymbol{\alpha})_j. \tag{4}$$

In writing this we have reference the potential to the pseudoscalar interaction strength $f_{\pi NN}$. The $\boldsymbol{\tau}_j, \boldsymbol{\tau}_j$ operators are iso-spin operators for the two nucleons, and the γ_5 matrix is given by

$$\gamma_5 = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix} \tag{5}$$

The $\gamma_5\boldsymbol{\alpha}$ product is sometimes written in terms of a spin operator according to

$$\boldsymbol{\Sigma} = \gamma_5\boldsymbol{\alpha} = \begin{bmatrix}\boldsymbol{\sigma} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\end{bmatrix} \tag{6}$$

3. Nonrelativistic reduction

The one-pion exchange potential is block diagonal, which allows for a direct nonrelativistic reduction

$$\hat{V}_{jk} = -\left(\frac{f_{\pi NN}}{m_\pi}\right)^2 (\boldsymbol{\tau}_k \cdot \boldsymbol{\tau}_j)\boldsymbol{\Sigma}_k \cdot \int \frac{\mathbf{k}\mathbf{k}}{(m_\pi c^2)^2 + \hbar^2 c^2 |\mathbf{k}|^2} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_k - \mathbf{r}_j)} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \cdot \boldsymbol{\Sigma}_j,$$

$$\Longrightarrow \hat{V}_{jk}^{(NR)} = -\left(\frac{f_{\pi NN}}{m_\pi}\right)^2 (\boldsymbol{\tau}_k \cdot \boldsymbol{\tau}_j)\boldsymbol{\sigma}_k \cdot \int \frac{\mathbf{k}\mathbf{k}}{(m_\pi c^2)^2 + \hbar^2 c^2 |\mathbf{k}|^2} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}_k - \mathbf{r}_j)} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \cdot \boldsymbol{\sigma}_j. \tag{7}$$

The lowest-order phonon–nuclear interaction involves block off-diagonal terms, which requires a unitary transformation to deal with.

3.1. Unitary transformation

A Foldy–Wouthuysen transformation in this case leads to

$$\hat{H}'_{int}(j,k) = \mathrm{e}^{i\hat{S}(j,k)} \hat{H}_{int}(j,k) \mathrm{e}^{-i\hat{S}(j,k)} \tag{8}$$

with

$$i\hat{S}(j,k) = \frac{1}{2mc}\left(\beta_j \boldsymbol{\alpha}_j \cdot \hat{\mathbf{p}}_j + \beta_k \boldsymbol{\alpha}_k \cdot \hat{\mathbf{p}}_k\right) + \cdots \tag{9}$$

The lowest-order contribution to the rotation is

$$\hat{H}'_{\text{int}}(j,k) \rightarrow \hat{H}_{\text{int}}(j,k) + \left[i\hat{S}(j,k), \hat{H}_{\text{int}}(j,k)\right]. \tag{10}$$

This leads to

$$\begin{aligned} \hat{H}'_{\text{int}}(j,k) \rightarrow\ & \frac{1}{2Mc}\left[\beta_j \boldsymbol{\alpha}_j + \beta_k \boldsymbol{\alpha}_k, \hat{V}_{jk}\right] \cdot \hat{\mathbf{P}} \\ & + \left[\frac{1}{2mc}\left(\beta_j \boldsymbol{\alpha}_j \cdot \hat{\mathbf{p}}_j + \beta_k \boldsymbol{\alpha}_k \cdot \hat{\mathbf{p}}_k\right), \frac{1}{2Mc}\left[\beta_j \boldsymbol{\alpha}_j + \beta_k \boldsymbol{\alpha}_k, \hat{V}_{jk}\right] \cdot \hat{\mathbf{P}}\right] \end{aligned} \tag{11}$$

Since there are no $++$ contributions from the first term, the leading order nonrelativistic interaction will come from the second term.

3.2. Evaluation

To evaluate this, the first step involves an expansion. We can write

$$\hat{H}'_{\text{int}}(j,k) \rightarrow \hat{H}_{\text{int}}(j,k)$$

$$+\frac{1}{4Mmc^2}\Big[\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j,\Big[\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}},\hat{V}_{jk}\Big]\Big]+\frac{1}{4Mmc^2}\Big[\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j,\Big[\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}},\hat{V}_{jk}\Big]\Big]$$

$$+\frac{1}{4Mmc^2}\Big[\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k,\Big[\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}},\hat{V}_{jk}\Big]\Big]+\frac{1}{4Mmc^2}\Big[\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k,\Big[\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}},\hat{V}_{jk}\Big]\Big]$$

$$= \hat{H}_{\text{int}}(j,k)$$

$$+\frac{1}{4Mmc^2}\Big(\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\hat{V}_{jk}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j+\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j$$

$$+\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\hat{V}_{jk}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}-\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j+\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j$$

$$+\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\hat{V}_{jk}-\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k+\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k$$

$$+\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\hat{V}_{jk}-\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}-\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k+\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\Big). \tag{12}$$

Since the potential is block diagonal, and since we will be taking the $++$ sector contribution as the nonrelativistic interaction, we can drop all terms that do not contribute to the $++$ sector. This leads to

$$\hat{H}'_{\text{int}}(j,k) \rightarrow$$

$$\frac{1}{4Mmc^2}\Big(\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\hat{V}_{jk}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j+\hat{V}_{jk}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j$$

$$+\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\hat{V}_{jk}-\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}-\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k+\hat{V}_{jk}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{P}}\beta_k\boldsymbol{\alpha}_k\cdot\hat{\mathbf{p}}_k\Big). \tag{13}$$

At this point it will be convenient to write the one-pion exchange potential as

$$\hat{V}_{jk} = \int(\gamma_5\boldsymbol{\alpha})_k\cdot\mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j\cdot\mathbf{k}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \tag{14}$$

with

$$\hat{v}_{jk} = -\left(\frac{f_{\pi NN}}{m_\pi}\right)^2(\boldsymbol{\tau}_k\cdot\boldsymbol{\tau}_j)\frac{1}{(m_\pi c^2)^2+\hbar^2c^2|\mathbf{k}|^2}e^{i\mathbf{k}\cdot(\mathbf{r}_k-\mathbf{r}_j)}. \tag{15}$$

Using this we can write

$$\hat{H}'_{\text{int}}(j,k) \rightarrow$$

$$\frac{1}{4Mmc^2}\int\Big(\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}(\gamma_5\boldsymbol{\alpha})_k\cdot\mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j\cdot\mathbf{k}-\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{p}}_j(\gamma_5\boldsymbol{\alpha})_k\cdot\mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j\cdot\mathbf{k}\beta_j\boldsymbol{\alpha}_j\cdot\hat{\mathbf{P}}$$

$$-\beta_j \boldsymbol{\alpha}_j \cdot \hat{\mathbf{P}}(\gamma_5\boldsymbol{\alpha})_k \cdot \mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j \cdot \mathbf{k}\beta_j\boldsymbol{\alpha}_j \cdot \hat{\mathbf{p}}_j + (\gamma_5\boldsymbol{\alpha})_k \cdot \mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j \cdot \mathbf{k}\beta_j\boldsymbol{\alpha}_j \cdot \hat{\mathbf{P}}\beta_j\boldsymbol{\alpha}_j \cdot \hat{\mathbf{p}}_j$$

$$+\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{p}}_k\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{P}}(\gamma_5\boldsymbol{\alpha})_k \cdot \mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j \cdot \mathbf{k} - \beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{p}}_k(\gamma_5\boldsymbol{\alpha})_k \cdot \mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j \cdot \mathbf{k}\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{P}}$$

$$-\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{P}}(\gamma_5\boldsymbol{\alpha})_k \cdot \mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j \cdot \mathbf{k}\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{p}}_k + (\gamma_5\boldsymbol{\alpha})_k \cdot \mathbf{k}\hat{v}_{jk}(\gamma_5\boldsymbol{\alpha})_j \cdot \mathbf{k}\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{P}}\beta_k\boldsymbol{\alpha}_k \cdot \hat{\mathbf{p}}_k\Bigg)\frac{d^3\mathbf{k}}{(2\pi)^3}. \qquad (16)$$

This can be expanded out in terms of 2×2 matrices to give

$$\hat{H}'_{\text{int}}(j,k) \ \rightarrow$$

$$\frac{1}{4Mmc^2}\int\Bigg(\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_j\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j$$

$$-\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_j\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_j$$

$$-\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_j\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_j$$

$$+\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_j$$

$$+\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_k\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_k\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j$$

$$-\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_k\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_k$$

$$-\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_k\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_k$$

$$+\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_k\hat{v}_{jk}\begin{bmatrix}\boldsymbol{\sigma}\cdot\mathbf{k} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}_j\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ -\boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{bmatrix}_k\begin{bmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{p}\\ -\boldsymbol{\sigma}\cdot\mathbf{p} & \mathbf{0}\end{bmatrix}_k\Bigg)\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}. \qquad (17)$$

Multiplying out leads to

$$\hat{H}'_{\text{int}}(j,k) \ \rightarrow$$

$$\frac{1}{4Mmc^2}\int\Bigg(-\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{k}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}_k\hat{v}_{jk}$$

$$+\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_j\cdot\mathbf{P}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k\hat{v}_{jk}$$

$$+\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k$$

$$-\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k$$

$$-\boldsymbol{\sigma}_j\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\boldsymbol{\sigma}\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k\hat{v}_{jk}$$

$$+\boldsymbol{\sigma}_j\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\boldsymbol{\sigma}_k\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{P}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k\hat{v}_{jk}$$

$$+\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k$$

$$-\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{k}\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\begin{bmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{bmatrix}_k\Bigg)\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}. \tag{18}$$

From this we can extract the nonrelativistic interaction

$$\left(\hat{H}'_{\mathrm{int}}(j,k)\right)_{NR}\rightarrow$$

$$\frac{1}{4Mmc^2}\int\Bigg(-\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{k}\hat{v}_{jk}+\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{k}\hat{v}_{jk}$$

$$+\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}-\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_k\cdot\mathbf{k}$$

$$-\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\boldsymbol{\sigma}_k\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{k}\hat{v}_{jk}+\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\boldsymbol{\sigma}_k\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{P}\hat{v}_{jk}$$

$$+\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k-\hat{v}_{jk}\boldsymbol{\sigma}_j\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{k}\boldsymbol{\sigma}_k\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\Bigg)\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}$$

$$=\frac{1}{4Mmc^2}\Bigg(-\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\boldsymbol{\sigma}_j\cdot\mathbf{P}\hat{V}_{jk}^{(NR)}+\boldsymbol{\sigma}_j\cdot\mathbf{p}_j\hat{V}_{jk}^{(NR)}\boldsymbol{\sigma}_j\cdot\mathbf{P}+\boldsymbol{\sigma}_j\cdot\mathbf{P}\hat{V}_{jk}^{(NR)}\boldsymbol{\sigma}_j\cdot\mathbf{p}_j-\hat{V}_{jk}^{(NR)}\boldsymbol{\sigma}_j\cdot\mathbf{P}\boldsymbol{\sigma}_j\cdot\mathbf{p}_j$$

$$-\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\boldsymbol{\sigma}_k\cdot\mathbf{P}\hat{V}_{jk}^{(NR)}+\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\hat{V}_{jk}^{(NR)}\boldsymbol{\sigma}_k\cdot\mathbf{P}+\boldsymbol{\sigma}_k\cdot\mathbf{P}\hat{V}_{jk}^{(NR)}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k-\hat{V}_{jk}^{(NR)}\boldsymbol{\sigma}_k\cdot\mathbf{P}\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\Bigg). \tag{19}$$

This can be written making use of commutators as

$$\left(\hat{H}'_{\text{int}}(j,k)\right)_{NR} = -\frac{1}{4Mmc^2}\left[\boldsymbol{\sigma}_j\cdot\mathbf{p}_j,\left[\boldsymbol{\sigma}_j\cdot\mathbf{P},\hat{V}_{jk}^{(NR)}\right]\right]-\frac{1}{4Mmc^2}\left[\boldsymbol{\sigma}_k\cdot\mathbf{p}_k,\left[\boldsymbol{\sigma}_k\cdot\mathbf{P},\hat{V}_{jk}^{(NR)}\right]\right]. \tag{20}$$

For the double commutators used here, we can write for general operators $\hat{A}$, $\hat{B}$ and $\hat{C}$

$$[\hat{A},[\hat{B},\hat{C}]] = [\hat{A},\hat{B}\hat{C}-\hat{C}\hat{B}] = \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}-\hat{B}\hat{C}\hat{A}+\hat{C}\hat{B}\hat{A}. \tag{21}$$

This result was presented previously at ICCF22.

4. Discussion

The lowest-order contribution to the phonon–nuclear coupling derived from the many-particle Dirac model is

$$\hat{H}_{\text{int}} = \frac{1}{2Mc}\sum_{j<k}\left[\beta_j\boldsymbol{\alpha}_j+\beta_k\boldsymbol{\alpha}_k,\hat{V}_{jk}\right]\cdot\hat{\mathbf{P}}. \tag{22}$$

For applications we require a nonrelativistic reduction, which in general is going to be complicated. For the special case of the relativistic one-pion exchange potential interaction based on pseudovector coupling, this reduces to

$$\left(\hat{H}'_{\text{int}}(j,k)\right)_{NR} = -\frac{1}{4Mmc^2}\left[\boldsymbol{\sigma}_j\cdot\mathbf{p}_j,\left[\boldsymbol{\sigma}_j\cdot\mathbf{P},\hat{V}_{jk}^{(NR)}\right]\right]-\frac{1}{4Mmc^2}\left[\boldsymbol{\sigma}_k\cdot\mathbf{p}_k,\left[\boldsymbol{\sigma}_k\cdot\mathbf{P},\hat{V}_{jk}^{(NR)}\right]\right]. \tag{23}$$

This is now in a form that can be used for specific calculations, as a way to develop estimates for the phonon–nuclear coupling matrix elements.

Note that if observations confirm the existence of this interaction, then it would follow that the pseudovector coupling model is more consistent with experiment than the pseudovector coupling model for the pion-nucleon interaction. This would be one of the few tests capable of unambiguously distinguishing between the two different models.

This interaction can be used in connection with phonon–nuclear matrix elements for specific models for anomalies. Most recently we have documented the reduction of the HD/^{3}He phonon–nuclear matrix element [11] which in our models is involved in the first step of the process for excess heat production in light water experiments. In future work we would like to make use of it for the calculation of the D_2/^{4}He phonon–nuclear coupling matrix element, which would be important for our models for excess heat production in PdD and in other heavy water experiments.

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